

# New class of gauge invariant solutions of Yang-Mills equations

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## Abstract

We consider field equation (system of equations) in pseudo-Euclidian space. This equation is invariant under orthogonal (of group  $O(p,q)$ ) coordinate transformations and invariant under gauge transformations of some group. We present new class of solutions of the Yang-Mills equations which corresponds to the solutions of the field equation.

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## Contents

<b>1</b>	<b>Dirac equation and Clifford algebras</b>	<b>2</b>
<b>2</b>	<b>Lie algebras in Clifford algebras</b>	<b>9</b>
<b>3</b>	<b>Projection operators and contractions in Clifford algebras</b>	<b>12</b>

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<b>4</b>	<b>Clifford field vectors and an algebra of <math>h</math>-forms.</b>	<b>15</b>
<b>5</b>	<b>Primitive field equation and its gauge symmetry</b>	<b>20</b>
<b>6</b>	<b>Solution of the primitive field equation</b>	<b>23</b>
<b>7</b>	<b>Solutions of Yang-Mills equations</b>	<b>29</b>

In physics field equations describe physical fields and (using quantization) elementary particles. The following equations are fundamental field equations: Maxwell's equations (1862), the Klein-Gordon-Fock equation (1926), the Dirac equation (1928), Yang-Mills equations (1954). These equations are considered in Minkowski space  $\mathbb{R}^{1,3}$ , they are invariant under Lorentz coordinate transformations. They are also invariant under certain gauge transformations.

For the last 60 years many particular solutions of Yang-Mills equations were discovered. Namely, monopoles (Wu, Yang, 1968 [1]), instantons (Belavin, Polyakov, Schwartz, Tyupkin, 1975 [2]), merons (de Alfaro, Fubini, Furlan, 1976 [3]) etc.<sup>1</sup>

In this paper we consider a new field equation (system of equations) (33) in the pseudo-Euclidian space  $\mathbb{R}^{p,q}$ , which is said to be a primitive field equation. We present new class of gauge-invariant solutions (47) of the Yang-Mills equations, which corresponds to the solutions of the primitive field equation (33).

A simple particular case (with spinor gauge symmetry) of the invented class of solutions of Yang-Mills equations was considered in [6].

Halmos symbol ■ denotes the end of proof of theorems.

## 1 Dirac equation and Clifford algebras

**Dirac equation and properties of  $\gamma$ -matrices.** Consider the Dirac equation for an electron in the Minkowski space  $\mathbb{R}^{1,3}$  with coordinates  $x^\mu$ ,  $\mu = 0, 1, 2, 3$  ( $\partial_\mu = \partial/\partial x^\mu$  – partial derivatives)

$$i\gamma^\mu(\partial_\mu\psi - ia_\mu\psi) - m\psi = 0, \quad (1)$$

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<sup>1</sup>See the review of Actor, 1979 [4] and the review of Zhdanov and Lagno, 2001 [5].

where  $\gamma^\mu$  are 4 complex square matrices of order 4 satisfying conditions

$$\partial_\mu \gamma^\nu = 0, \quad (2)$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} I, \quad (3)$$

where  $\eta = \|\eta^{\mu\nu}\| = \text{diag}(1, -1, -1, -1)$ ,  $I$  is the identity matrix of order 4,  $a_\mu = a_\mu(x)$  is a covector potential of electromagnetic field,  $\psi = \psi(x)$  is a Dirac spinor (column of four complex functions),  $i$  is the imaginary unit,  $m$  is a real number (mass of electron).

In the theory of the Dirac equation it is assumed that we have a fixed set of matrices  $\gamma^\mu$  that satisfy conditions (2), (3) and the condition<sup>2</sup> for Hermitian conjugated matrices

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0. \quad (4)$$

Matrices  $\gamma^\mu$  satisfying conditions (2), (3), (4) are defined up to a similarity transformation with a unitary matrix  $U \in U(4)$ , i.e. matrices

$$\acute{\gamma}^\mu = U^{-1} \gamma^\mu U \quad (5)$$

satisfy the same conditions (2), (3), (4).

In particular, matrices  $\gamma^0, \gamma^1, \gamma^2, \gamma^3$  in the Dirac representation satisfy these conditions and the matrix  $\gamma^0$  is diagonal

$$\gamma^0 = \text{diag}(1, 1, -1, -1).$$

This matrix  $\gamma^0$  changes under unitary transformation (5).

Denote

$$\beta = \text{diag}(1, 1, -1, -1)$$

and consider Lie group  $SU(2, 2)$  of special pseudo-unitary matrices and its real Lie algebra  $\mathfrak{su}(2, 2)$  (see [7])

$$\begin{aligned} SU(2, 2) &= \{S \in \text{Mat}(4, \mathbb{C}) : S^\dagger \beta S = \beta, \det S = 1\}, \\ \mathfrak{su}(2, 2) &= \{s \in \text{Mat}(4, \mathbb{C}) : \beta s^\dagger \beta = -s, \text{tr } s = 0\}, \end{aligned}$$

where  $\text{Mat}(4, \mathbb{C})$  is the algebra of complex matrices of order 4. Dirac gamma matrices  $\gamma^\mu$  satisfy (4), therefore

$$i\gamma^\mu \in \mathfrak{su}(2, 2). \quad (6)$$

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<sup>2</sup>This condition requires when we consider bilinear covariants of the Dirac spinors.

We may consider conditions (2), (3) together with condition (6) and allow a similarity transformation

$$i\gamma^\mu \rightarrow i\dot{\gamma}^\mu = S^{-1}i\gamma^\mu S \quad (7)$$

with matrix  $S \in \text{SU}(2, 2)$ , which preserves (2), (3) and (6).

If we consider conditions (2), (3) as equations for matrices  $\gamma^\mu$  with condition (6), then we can consider transformation (7) as global symmetry (it does not depend on  $x \in \mathbb{R}^{1,3}$ ) of this system of equations.

Now we change equations (2) and obtain system of equations with local (gauge) symmetry with respect to the pseudo-unitary group.

Namely, consider the following system of equations [8]:

$$\partial_\mu \gamma^\nu - [C_\mu, \gamma^\nu] = 0, \quad (8)$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} I, \quad (9)$$

where  $i\gamma^\mu = i\gamma^\mu(x)$  and  $C_\mu = C_\mu(x)$  are functions of  $x \in \mathbb{R}^{1,3}$  with values in the Lie algebra  $\text{su}(2, 2)$ . The system of equations (8), (9) is invariant under a local (gauge) transformation

$$i\gamma^\mu \rightarrow i\dot{\gamma}^\mu = S^{-1}i\gamma^\mu S, \quad (10)$$

$$C_\mu \rightarrow \dot{C}_\mu = S^{-1}C_\mu S - S^{-1}\partial_\mu S, \quad (11)$$

where the matrix  $S = S(x)$  is a function of  $x \in \mathbb{R}^{1,3}$  with values in the Lie group  $\text{SU}(2, 2)$ .

We consider system of equations (8), (9) as a new field equation (system of equations). We call this equation a *primitive field equation*. Let us analyze this equation in pseudo-Euclidian spaces. We use a formalism of Clifford algebras because, in our opinion, this formalism is the most convenient for this task.

**Clifford algebra  $\mathcal{C}^\mathbb{F}(p, q)$ .** Let  $E$  be a vector space over the field  $\mathbb{F}$  of real numbers  $\mathbb{R}$  or complex numbers  $\mathbb{C}$ . Let  $n$  be a natural number and the dimension of vector space  $E$  is equal to  $\dim E = 2^n$ . We consider a basis

$$e, e^a, e^{a_1 a_2}, \dots, e^{1 \dots n}, \quad \text{where } a_1 < a_2 < \dots, \quad (12)$$

of  $E$ , numerated by ordered multi-indices of lengths from 0 to  $n$ . The indices  $a, a_1, a_2, \dots$  range from 1 to  $n$ .

We introduce the operation of *Clifford multiplication*  $U, V \rightarrow UV$  on  $E$  such that

- 1) (distributivity and linearity)  
for all  $U, V, W \in E$  and  $\alpha, \beta \in \mathbb{F}$

$$U(\alpha V + \beta W) = \alpha UV + \beta UW, (\alpha U + \beta V)W = \alpha UW + \beta VW,$$

- 2) (associativity) for all  $U, V, W \in E$

$$(UV)W = U(VW),$$

- 3) (unitality) for all  $U \in E$

$$Ue = eU = U,$$

- 4) for all  $a, b = 1, \dots, n$

$$e^a e^b + e^b e^a = 2\eta^{ab}e, \quad (13)$$

where  $\eta = \|\eta^{ab}\|$  is the diagonal matrix of order  $n$

$$\eta = \|\eta^{ab}\| = \text{diag}(1, \dots, 1, -1, \dots, -1) \quad (14)$$

with  $p$  copies of 1 and  $q$  copies of  $-1$  on the diagonal.

- 5) for all  $1 \leq a_1 < \dots < a_k \leq n$

$$e^{a_1} \dots e^{a_k} = e^{a_1 \dots a_k}.$$

These rules define an algebra, which is called a *Clifford algebra* and is denoted by  $\mathcal{C}^{\mathbb{R}}(p, q)$  in the case of real field and by  $\mathcal{C}^{\mathbb{C}}(p, q) = \mathcal{C}(p, q)$  in the case of the complex field [9]. Note that

$$\mathcal{C}^{\mathbb{R}}(p, q) \subset \mathcal{C}(p, q).$$

When our argumentation is applicable to both cases, we write  $\mathcal{C}^{\mathbb{F}}(p, q)$ , implying that  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{F} = \mathbb{C}$ .

The element  $e$  is called the Clifford algebra *identity*. The elements  $e^a$  are called Clifford algebra *generators*. The pair  $(p, q)$  is called the *signature* of the Clifford algebra  $\mathcal{C}^{\mathbb{F}}(p, q)$ . Note that in literature the number  $p - q$  also is called the signature.

The number  $n = p + q$  is called the *dimension of Clifford algebra*  $\mathcal{C}^\mathbb{F}(p, q)$  (but the dimension of Clifford algebra as a vector space equals  $2^n$ ).

So, we have (see properties 1-5) associative noncommutative (for  $n \geq 2$ ) unital algebra with the defining relations (13).

Any element  $U$  of the Clifford algebra  $\mathcal{C}^\mathbb{F}(p, q)$  can be expanded in the basis (12):

$$U = ue + u_a e^a + \sum_{a_1 < a_2} u_{a_1 a_2} e^{a_1 a_2} + \dots + u_{1\dots n} e^{1\dots n}, \quad (15)$$

where  $u, u_a, u_{a_1 a_2}, \dots, u_{1\dots n}$  are real or complex numbers (in the respective cases  $\mathcal{C}^\mathbb{R}(p, q)$  or  $\mathcal{C}(p, q)$ ).

Vector (real or complex) subspaces spanned by basis elements  $e^{a_1 \dots a_k}$  labeled by ordered multi-indices of length  $k$  are denoted by  $\mathcal{C}_k^\mathbb{F}(p, q)$ ,  $k = 0, \dots, n$ . Elements of the subspace  $\mathcal{C}_k^\mathbb{F}(p, q)$  are called *elements of rank  $k$* . We have

$$\mathcal{C}^\mathbb{F}(p, q) = \mathcal{C}_0^\mathbb{F}(p, q) \oplus \dots \oplus \mathcal{C}_n^\mathbb{F}(p, q) = \mathcal{C}_{\text{Even}}^\mathbb{F}(p, q) \oplus \mathcal{C}_{\text{Odd}}^\mathbb{F}(p, q),$$

where

$$\begin{aligned} \mathcal{C}_{\text{Even}}^\mathbb{F}(p, q) &= \mathcal{C}_0^\mathbb{F}(p, q) \oplus \mathcal{C}_2^\mathbb{F}(p, q) \oplus \dots, \\ \mathcal{C}_{\text{Odd}}^\mathbb{F}(p, q) &= \mathcal{C}_1^\mathbb{F}(p, q) \oplus \mathcal{C}_3^\mathbb{F}(p, q) \oplus \dots, \end{aligned}$$

$$\dim \mathcal{C}_k^\mathbb{F}(p, q) = C_n^k, \quad \dim \mathcal{C}_{\text{Even}}^\mathbb{F}(p, q) = \dim \mathcal{C}_{\text{Odd}}^\mathbb{F}(p, q) = 2^{n-1}$$

and  $C_n^k = \frac{n!}{k!(n-k)!}$  are binomial coefficients. The Clifford algebra is a  $Z_2$ -graded algebra (superalgebra). Elements of the subspace  $\mathcal{C}_{\text{Even}}^\mathbb{F}(p, q)$  are called *even* elements and elements of the subspace  $\mathcal{C}_{\text{Odd}}^\mathbb{F}(p, q)$  are called *odd* elements of Clifford algebra.

We introduce the operations of projection onto subspaces of rank- $k$  elements:

$$\pi_k : \mathcal{C}^\mathbb{F}(p, q) \rightarrow \mathcal{C}_k^\mathbb{F}(p, q), \quad k = 0, 1, \dots, n.$$

For element  $U \in \mathcal{C}^\mathbb{F}(p, q)$  (15) we have

$$\pi_k(U) = \sum_{a_1 < \dots < a_k}^k u_{a_1 \dots a_k} e^{a_1 \dots a_k} \in \mathcal{C}_k^\mathbb{F}(p, q). \quad (16)$$

The Clifford algebra  $\mathcal{C}_k^\mathbb{F}(p, q)$  has the following center

$$\text{cen} \mathcal{C}^\mathbb{F}(p, q) = \begin{cases} \mathcal{C}_0^\mathbb{F}(p, q), & \text{if } n \text{ is even;} \\ \mathcal{C}_0^\mathbb{F}(p, q) \oplus \mathcal{C}_n^\mathbb{F}(p, q) & \text{if } n \text{ is odd.} \end{cases}$$

**Pseudo-euclidian space  $\mathbb{R}^{p,q}$  and changes of coordinates.** Let  $p, q$  be nonnegative integers and  $n = p + q$ . We denote an  $n$ -dimensional pseudo-Euclidian space of signature  $(p, q)$  with Cartesian coordinates  $x^\mu$ ,  $\mu = 1, \dots, n$  by  $\mathbb{R}^{p,q}$ . Tensor indices corresponding to the coordinates are denoted by small Greek letters. The metric tensor of pseudo-Euclidian space  $\mathbb{R}^{p,q}$  is given by diagonal matrix of order  $n$

$$\eta = \|\eta_{\mu\nu}\| = \|\eta^{\mu\nu}\| = \text{diag}(1, \dots, 1, -1, \dots, -1) \quad (17)$$

with  $p$  copies of 1 and  $q$  copies of  $-1$  on the diagonal.

In  $\mathbb{R}^{p,q}$  we deal with linear coordinate transformations<sup>3</sup>

$$x^\mu \rightarrow \acute{x}^\mu = p_\nu^\mu x^\nu, \quad (18)$$

preserving the metric tensor. So, real numbers  $p_\nu^\mu$  satisfy relations

$$p_\alpha^\mu p_\beta^\nu \eta^{\alpha\beta} = \eta^{\mu\nu}, \quad p_\alpha^\mu p_\beta^\nu \eta_{\mu\nu} = \eta_{\alpha\beta}. \quad (19)$$

In matrix formalism we can write

$$P^T \eta P = \eta, \quad P \eta P^T = \eta,$$

where  $T$  is the matrix transposition and the matrix  $P = \|p_\nu^\mu\|$  is from the pseudo-orthogonal group

$$O(p, q) = \{P \in \text{Mat}(n, \mathbb{R}) : P^T \eta P = \eta\}.$$

We denote the set of  $(r, s)$  tensor fields (of rank  $r + s$ ) of pseudo-Euclidian space  $\mathbb{R}^{p,q}$  by  $T_s^r$ . Real and complex tensor field  $u \in T_s^r$  has components  $u_{\nu_1 \dots \nu_s}^{\mu_1 \dots \mu_r}$  in coordinates  $x^\mu$ . These components are smooth functions  $\mathbb{R}^{p,q} \rightarrow \mathbb{F}$ , where  $\mathbb{F}$  is the field of real numbers  $\mathbb{R}$  or complex numbers  $\mathbb{C}$ . In all considerations of this work it is sufficient that all functions of  $x \in \mathbb{R}^{p,q}$  have continuous partial derivatives up to the second order.

**Functions with values in Clifford algebra.** Further we consider functions  $\mathbb{R}^{p,q} \rightarrow \mathcal{C}(p, q)$  with values in Clifford algebra. We assume that the basis elements (12) does not depend on the points  $x \in \mathbb{R}^{p,q}$  i.e.

$$\partial_\mu e^a = 0, \quad \forall \mu, a = 1, \dots, n,$$

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<sup>3</sup> We use Einstein summation convention. According to this convention, when an index variable appears twice in a single term it implies summation of that term over all the values of the index (one of indices is upper - contravariant, and the second is lower - covariant). For example  $p_\nu^\mu x^\nu = \sum_{\nu=1}^n p_\nu^\mu x^\nu$ .

where  $\partial_\mu = \partial/\partial x^\mu$  are partial derivatives. The coefficients in the basis expansion of the Clifford algebra element may depend on  $x \in \mathbb{R}^{p,q}$ . For example, in the basis expansion (16) we have

$$U = \overset{k}{U} \overset{k}{U}(x), \quad u_{a_1 \dots a_k} = u_{a_1 \dots a_k}(x), \quad x \in \mathbb{R}^{p,q},$$

where the coefficients of the Clifford algebra element  $\mathcal{C}^\mathbb{F}(p, q)$  are smooth functions of  $x$ .

In the present paper we also consider the functions with values in Lie algebras generated by the Clifford algebra (see p. 11).

**Tensor fields with values in Clifford algebra.** Tensor at the point  $x \in \mathbb{R}^{p,q}$  with values in Clifford algebra is a mathematical object that belongs to the tensor product of the tensor algebra and Clifford algebra.

If a  $(r, s)$  tensor field in  $\mathbb{R}^{p,q}$  has components  $u_{\nu_1 \dots \nu_s}^{\mu_1 \dots \mu_r} = u_{\nu_1 \dots \nu_s}^{\mu_1 \dots \mu_r}(x)$  in Cartesian coordinates  $x^\mu$ , then these components are considered as functions  $\mathbb{R}^{p,q} \rightarrow \mathbb{F}$ . These functions transform by the standard tensor transformation law.

Components  $U_{\nu_1 \dots \nu_s}^{\mu_1 \dots \mu_r}$  of tensor fields with values in Clifford algebra  $\mathcal{C}^\mathbb{F}(p, q)$  are considered as functions  $\mathbb{R}^{p,q} \rightarrow \mathcal{C}^\mathbb{F}(p, q)$  that transform under changes of coordinates by the standard tensor transformation law.

We use the following notation for tensor fields with values in Clifford algebra:  $U_{\nu_1 \dots \nu_s}^{\mu_1 \dots \mu_r} \in \mathcal{C}(p, q)T_s^r$  or  $U \in \mathcal{C}(p, q)T_s^r$ . In this notation the letter T means that this object is a tensor field. In particular, for scalar functions  $U : \mathbb{R}^{p,q} \rightarrow \mathcal{C}^\mathbb{F}(p, q)$  we use the notation  $U \in \mathcal{C}^\mathbb{F}(p, q)T$ .

Example. If we consider a tensor field  $U_\nu^\mu \in \mathcal{C}(p, q)T_1^1$  with values in Clifford algebra, then we can write

$$U = u_\nu^\mu e + u_{\nu a}^\mu e^a + \sum_{a_1 < a_2} u_{\nu a_1 a_2}^\mu e^{a_1 a_2} + \dots + u_{\nu 1 \dots n}^\mu e^{1 \dots n},$$

where  $u_\nu^\mu, u_{\nu a}^\mu, u_{\nu a_1 a_2}^\mu, \dots, u_{\nu 1 \dots n}^\mu$  are real (in the case of  $\mathcal{C}^\mathbb{R}(p, q)$ ) or complex (in the case of  $\mathcal{C}^\mathbb{C}(p, q)$ ) tensor fields  $T_1^1$ .

In the present paper we also consider tensor fields with values in Lie algebras and scalar fields with values in Lie groups (see p.11).



## 2 Lie algebras in Clifford algebras

Let consider commutator (Lie bracket)

$$[U, V] = UV - VU$$

of Clifford algebra elements  $U, V \in \mathcal{C}(p, q)$ . This operation satisfy the Jacobi identity

$$[[U, V], W] + [[V, W], U] + [[W, U], V] = 0, \quad \forall U, V, W \in \mathcal{C}(p, q).$$

Therefore, Clifford algebra  $\mathcal{C}(p, q)$  can be considered as a Lie algebra with respect to the commutator. We can consider vector subspaces  $L \subset \mathcal{C}(p, q)$  of Clifford algebra that closed under commutator i.e. with condition: if  $U, V \in L$  then  $[U, V] \in L$ . These subspaces are Lie algebras (generated by Clifford algebra). Primarily we are interested in Lie algebras that are direct sums (as vector spaces) of subspaces of Clifford algebra elements of fixed ranks [10].

With the help of the operator  $\pi_0 : \mathcal{C}^{\mathbb{F}}(p, q) \rightarrow \mathcal{C}_0^{\mathbb{F}}(p, q)$  (see (16)) we define operation of *Clifford algebra trace*  $\text{Tr} : \mathcal{C}^{\mathbb{F}} \rightarrow \mathbb{F}$

$$\text{Tr}(U) = \pi_0(U)|_{e \rightarrow 1}, \quad \forall U \in \mathcal{C}^{\mathbb{F}}(p, q).$$

**Theorem 1** *In Clifford algebra  $\mathcal{C}^{\mathbb{F}}(p, q)$  we have*

$$\text{Tr}([U, V]) = 0, \quad \forall U, V \in \mathcal{C}^{\mathbb{F}}(p, q)$$

*in the case of even  $n$  and*

$$\text{Tr}([U, V]) = 0, \quad \pi_n([U, V]) = 0, \quad \forall U, V \in \mathcal{C}^{\mathbb{F}}(p, q)$$

*in the case of odd  $n$ .*

**Proof.** This statement follows from the formulas for commutators of Clifford algebra elements of fixed ranks (see [9]). ■

Consider the set of Clifford algebra elements with zero projection onto Clifford algebra center

$$\mathcal{C}_{\mathbb{S}}(p, q) = \mathcal{C}(p, q) \setminus \text{cen}\mathcal{C}(p, q).$$

**Theorem 2** *The set  $\mathcal{C}_{\otimes}(p, q)$  is a Lie algebra with respect to the commutator  $[A, B] = AB - BA$ .*

**Proof.** See the previous theorem. ■

**Theorem 3** *Let  $F = F(x)$  be function with values in Lie algebra  $\mathcal{C}_{\otimes}(p, q)$ . Then the partial derivatives  $\partial_{\mu}F$  are functions (components of a covariant vector field) with values in the same Lie algebra  $\mathcal{C}_{\otimes}(p, q)$ .*

**Proof.** If  $n$  is even, then the function  $F = F(x)$  can be written as basis expansion (12)

$$F = f_a e^a + \sum_{a_1 < a_2} f_{a_1 a_2} e^{a_1 a_2} + \dots + f_{1\dots n} e^{1\dots n}, \quad (20)$$

Since  $\text{Tr } F = 0$ , then the first term  $f e$  is absent. We assume that Clifford algebra generators  $e^a$  do not depend on  $x \in \mathbb{R}^{p,q}$ . So

$$\partial_{\mu} e^a = 0, \quad \forall \mu, a = 1, \dots, n$$

and

$$\partial_{\mu} F = (\partial_{\mu} f_a) e^a + \sum_{a_1 < a_2} (\partial_{\mu} f_{a_1 a_2}) e^{a_1 a_2} + \dots + (\partial_{\mu} f_{1\dots n}) e^{1\dots n}.$$

We obtain

$$\text{Tr } F = 0 \implies \text{Tr}(\partial_{\mu} F) = 0,$$

i.e.  $\partial_{\mu} F \in \mathcal{C}_{\otimes}(p, q)$ .

If  $n$  is odd, then the function  $F = F(x) \in \mathcal{C}_{\otimes}(p, q)$  can be written as basis expansion (20) without the first term  $f e$  and without the last term  $f_{1\dots n} e^{1\dots n}$ . We obtain  $\partial_{\mu} F \in \mathcal{C}_{\otimes}(p, q)$  again. ■

The following subspaces of Clifford algebra are Lie algebras with respect to the commutator:

$$\mathcal{C}_2(p, q), \mathcal{C}_1(p, q) \oplus \mathcal{C}_2(p, q), \mathcal{C}_2(p, q) \oplus \mathcal{C}_3(p, q), \mathcal{C}_0(p, q), \text{cen}\mathcal{C}(p, q), \mathcal{C}_{\otimes}(p, q).$$

In the first section we have considered gamma-matrices in Dirac representation (which are used in the Dirac equation for an electron) and found

that  $i\gamma^\mu \in \mathfrak{su}(2, 2)$ . In Clifford algebra  $\mathcal{C}(p, q)$  the following Lie algebra is analogue<sup>4</sup> of the Lie algebra  $\mathfrak{su}(2, 2)$ :

$$\mathfrak{w}(\mathcal{C}(p, q)) = \{U \in \bigoplus_{k=1}^{\acute{n}} i^{\frac{k(k-1)}{2}+1} \mathcal{C}_k^{\mathbb{R}}(p, q)\},$$

where  $\acute{n} = n$  in the case of even  $n$  and  $\acute{n} = n - 1$  in the case of odd  $n$ . In other words:

$$\mathfrak{w}(\mathcal{C}(p, q)) = i\mathcal{C}_1^{\mathbb{R}}(p, q) \oplus \mathcal{C}_2^{\mathbb{R}}(p, q) \oplus \mathcal{C}_3^{\mathbb{R}}(p, q) \oplus i\mathcal{C}_4^{\mathbb{R}}(p, q) \oplus \dots \oplus a_{\acute{n}} \mathcal{C}_{\acute{n}}^{\mathbb{R}}(p, q),$$

where  $a_k = i$  if  $k = 0, 1 \bmod 4$  and  $a_k = 1$  if  $k = 2, 3 \bmod 4$ .

We are interested in Lie subalgebras of this Lie algebra. As we will see, Lie algebra  $\mathcal{C}_2^{\mathbb{R}}(p, q)$  plays a very important role in field theory equations. Other important Lie algebras contain Lie subalgebra  $i\mathcal{C}_1^{\mathbb{R}}(p, q) \oplus \mathcal{C}_2^{\mathbb{R}}(p, q)$ :

- For  $n \geq 2$

$$i\mathcal{C}_1^{\mathbb{R}}(p, q) \oplus \mathcal{C}_2^{\mathbb{R}}(p, q). \quad (21)$$

- For  $n \geq 6$

$$i\mathcal{C}_1^{\mathbb{R}}(p, q) \oplus \mathcal{C}_2^{\mathbb{R}}(p, q) \oplus a_{\acute{n}-1} \mathcal{C}_{\acute{n}-1}^{\mathbb{R}}(p, q) \oplus a_{\acute{n}} \mathcal{C}_{\acute{n}}^{\mathbb{R}}(p, q), \quad (22)$$

where  $\acute{n} = n$  for even  $n$  and  $\acute{n} = n - 1$  for odd  $n$ .

- For  $n \geq 8$

$$\begin{aligned} & i\mathcal{C}_1^{\mathbb{R}}(p, q) \oplus \mathcal{C}_2^{\mathbb{R}}(p, q) \oplus i\mathcal{C}_5^{\mathbb{R}}(p, q) \oplus \mathcal{C}_6^{\mathbb{R}}(p, q) \\ & \oplus i\mathcal{C}_9^{\mathbb{R}}(p, q) \oplus \mathcal{C}_{10}^{\mathbb{R}}(p, q) \oplus \dots \oplus a_r \mathcal{C}_r^{\mathbb{R}}(p, q), \end{aligned} \quad (23)$$

where

$$\begin{aligned} r &= n - 2, \text{ if } n = 0 \bmod 4, & r &= n - 3, \text{ if } n = 1 \bmod 4, \\ r &= n, \text{ if } n = 2 \bmod 4, & r &= n - 1, \text{ if } n = 3 \bmod 4. \end{aligned}$$

We consider pinor groups as the following sets of Clifford algebra elements:

$$\text{Pin}(p, q) = \{S \in \mathcal{C}_{\text{Even}}^{\mathbb{R}}(p, q) \text{ or } S \in \mathcal{C}_{\text{Odd}}^{\mathbb{R}}(p, q) : S \sim S = \pm e, S^{-1} e^a S \in \mathcal{C}_1^{\mathbb{R}}(p, q)\},$$

where linear operation  $\sim: \mathcal{C}_k(p, q) \rightarrow \mathcal{C}_k(p, q)$ ,  $k = 0, 1, \dots, n$  is called *reversion*. This operation reverses the order of generators in products:

$$(e^{a_1} \dots e^{a_k})^\sim = e^{a_k} \dots e^{a_1}.$$

Note that the set of rank 2 Clifford algebra elements  $\mathcal{C}_2^{\mathbb{R}}(p, q)$  is closed w.r.t. commutator and hence generates a Lie algebra. The Lie algebra  $\mathcal{C}_2^{\mathbb{R}}(p, q) \subset \mathfrak{w}(\mathcal{C}(p, q))$  is a real Lie algebra of the Lie group  $\text{Pin}(p, q)$  (see [11]).

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<sup>4</sup>See [8] and [9].

### 3 Projection operators and contractions in Clifford algebras

Consider operations of projection (16) onto subspaces  $\mathcal{C}_k(p, q)$  of Clifford algebra elements of rank  $k$ . For  $U \in \mathcal{C}(p, q)$  we have

$$\pi_k(U) \in \mathcal{C}_k(p, q).$$

The following sum is called a *generator contraction* of an arbitrary Clifford algebra element  $U \in \mathcal{C}(p, q)$ :

$$F(U) = e^a U e_a, \quad (24)$$

where  $e_a = \eta_{ab} e^b$ . We use notations  $F^0(U) = U$ ,  $F^1(U) = F(U)$ ,  $F^2(U) = F(F(U))$ , etc.

The operator  $F^l : \mathcal{C}(p, q) \rightarrow \mathcal{C}(p, q)$  is called the *generator contraction of order  $l$* . Note that

$$F^l : \mathcal{C}_k(p, q) \rightarrow \mathcal{C}_k(p, q), \quad \forall k, l = 0, 1, \dots, n.$$

According to the theorem on generator contraction ([9], [12]) we have

$$F(U) = \sum_{k=0}^n \lambda_k \pi_k(U), \quad \text{where } \lambda_k = (-1)^k (n - 2k). \quad (25)$$

**Theorem 4** Consider an arbitrary Clifford algebra element  $U \in \mathcal{C}(p, q)$ ,  $n = p + q$ . Then we have

$$\pi_k(U) = \sum_{l=0}^n b_{kl} F^l(U), \quad \text{if } n \text{ is even}, \quad (26)$$

$$\pi_{k,n-k}(U) = \sum_{l=0}^{\frac{n-1}{2}} g_{kl} F^l(U), \quad \text{if } n \text{ is odd}, \quad (27)$$

where  $B = ||b_{kl}||$  is inverse of matrix  $A_{(n+1) \times (n+1)} = ||a_{kl}||$ ,  $a_{kl} = (\lambda_{l-1})^{k-1}$ ,  $G = ||g_{kl}||$  is inverse of matrix  $D_{\frac{n+1}{2} \times \frac{n+1}{2}} = ||d_{kl}||$ ,  $d_{kl} = (\lambda_{l-1})^{k-1}$  and  $\lambda_k = (-1)^k (n - 2k)$ .

**Proof.** We have

$$F^l(U) = \sum_{k=0}^n (\lambda_k)^l \pi_k(U),$$

then

$$\begin{pmatrix} F^0(U) \\ F^1(U) \\ \dots \\ F^n(U) \end{pmatrix} = A \begin{pmatrix} \pi_0(U) \\ \pi_1(U) \\ \dots \\ \pi_n(U) \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_0 & \lambda_1 & \dots & \lambda_n \\ \dots & \dots & \dots & \dots \\ (\lambda_0)^n & (\lambda_1)^n & \dots & (\lambda_n)^n \end{pmatrix}.$$

Matrix  $A$  is a Vandermonde matrix. Its determinant equals

$$\det A = \prod_{0 \leq i < j \leq n} (\lambda_j - \lambda_i).$$

In the case of even  $n$  we have

$$\lambda_k = -\lambda_{n-k},$$

because  $\lambda_{n-k} = (-1)^{n-k}(n-2(n-k)) = (-1)^k(2k-n) = -\lambda_k$ . In particular,  $\lambda_{\frac{n}{2}} = 0$ . It is easy to see that all  $\lambda_k$  are different in the case of even  $n$ , and Vandermonde matrix is invertible. Denote the inverse matrix by  $B = ||b_{ij}||$ :

$$\begin{pmatrix} \pi_0(U) \\ \pi_1(U) \\ \dots \\ \pi_n(U) \end{pmatrix} = \begin{pmatrix} b_{00} & b_{01} & \dots & b_{0n} \\ b_{10} & b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots & \dots \\ b_{n0} & b_{n1} & \dots & b_{nn} \end{pmatrix} \begin{pmatrix} F^0(U) \\ F^1(U) \\ \dots \\ F^n(U) \end{pmatrix}.$$

There exists the explicit formula for inverse of Vandermonde matrix but we do not use it.

In the case of odd  $n$  we have

$$\lambda_k = \lambda_{n-k},$$

and hence Vandermonde matrix is singular and projection operations do not expressed through contractions.

However, consider operations of projection onto subspaces  $\mathcal{C}_k(p, q) \oplus \mathcal{C}_{n-k}(p, q)$ :

$$\pi_{k,n-k}(U) = \pi_k(U) + \pi_{n-k}(U).$$

We have

$$\begin{pmatrix} F^0(U) \\ F^1(U) \\ \dots \\ F^{\frac{n-1}{2}}(U) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_0 & \lambda_1 & \dots & \lambda_{\frac{n-1}{2}} \\ \dots & \dots & \dots & \dots \\ (\lambda_0)^{\frac{n-1}{2}} & (\lambda_1)^{\frac{n-1}{2}} & \dots & (\lambda_{\frac{n-1}{2}})^{\frac{n-1}{2}} \end{pmatrix} \begin{pmatrix} \pi_{0,n}(U) \\ \pi_{1,n-1}(U) \\ \dots \\ \pi_{\frac{n-1}{2}, \frac{n+1}{2}}(U) \end{pmatrix}.$$

We denote the invertible matrix from the last formula by  $D$  and inverse of  $D$  by  $G = ||g_{ij}||$ .

We obtain the relation between projection operations and contractions in the following form:

$$\begin{pmatrix} \pi_{0,n}(U) \\ \pi_{1,n-1}(U) \\ \dots \\ \pi_{\frac{n-1}{2}, \frac{n+1}{2}}(U) \end{pmatrix} = \begin{pmatrix} g_{00} & g_{01} & \dots & g_{0\frac{n-1}{2}} \\ g_{10} & g_{11} & \dots & g_{1\frac{n-1}{2}} \\ \dots & \dots & \dots & \dots \\ g_{\frac{n-1}{2}0} & g_{\frac{n-1}{2}1} & \dots & g_{\frac{n-1}{2}\frac{n-1}{2}} \end{pmatrix} \begin{pmatrix} F^0(U) \\ F^1(U) \\ \dots \\ F^{\frac{n-1}{2}}(U) \end{pmatrix}.$$

■

So, in the case of even  $n$  operations of projection of Clifford algebra elements  $U \in \mathcal{C}(p, q)$  is uniquely expressed through contractions (of order not more than  $n$ ) of element  $U$ . Note that we can use these formulas as the definition of operations of projection onto subspaces of fixed ranks.

Let's give some examples. In the case of  $n = 2$  we have

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 4 & 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{8} \\ 1 & 0 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{8} \end{pmatrix},$$

i.e.

$$\begin{aligned} F^0(U) &= U = \pi_0(U) + \pi_1(U) + \pi_2(U), \\ F^1(U) &= 2\pi_0(U) - 2\pi_2(U), \\ F^2(U) &= 4\pi_0(U) + 4\pi_2(U). \end{aligned}$$

and

$$\begin{aligned} \pi_0(U) &= \frac{1}{4}F^1(U) + \frac{1}{8}F^2(U) = \frac{1}{4}e^a U e_a + \frac{1}{8}e^a e^b U e_b e_a, \\ \pi_1(U) &= F^0(U) - \frac{1}{4}F^2(U) = U - \frac{1}{4}e^a e^b U e_a e_b, \\ \pi_2(U) &= -\frac{1}{4}F^1(U) + \frac{1}{8}F^2(U) = -\frac{1}{4}e^a U e_a + \frac{1}{8}e^a e^b U e_b e_a. \end{aligned}$$

In the case of  $n = 4$  we have

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & 0 & 2 & -4 \\ 16 & 4 & 0 & 4 & 16 \\ 64 & -8 & 0 & 8 & -64 \\ 256 & 16 & 0 & 16 & 256 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\frac{1}{24} & -\frac{1}{96} & \frac{1}{96} & \frac{1}{384} \\ 0 & -\frac{1}{3} & \frac{1}{6} & \frac{1}{48} & -\frac{1}{96} \\ 1 & 0 & -\frac{5}{16} & 0 & \frac{1}{64} \\ 0 & \frac{1}{3} & \frac{1}{6} & -\frac{1}{48} & -\frac{1}{96} \\ 0 & \frac{1}{24} & -\frac{1}{96} & -\frac{1}{96} & \frac{1}{384} \end{pmatrix}. \quad (28)$$

In the case of odd dimension  $n = 3$  we have

$$\begin{aligned} F^0(U) &= U = \pi_0(U) + \pi_1(U) + \pi_2(U) + \pi_3(U), \\ F^1(U) &= 3\pi_0(U) - \pi_1(U) - \pi_2(U) + 3\pi_3(U), \\ F^2(U) &= 9\pi_0(U) + \pi_1(U) + \pi_2(U) + 9\pi_3(U), \\ F^3(U) &= 27\pi_0(U) - \pi_1(U) - \pi_2(U) + 27\pi_3(U). \end{aligned}$$

Matrix of this system of equations is singular. But we can consider expressions

$$\pi_{03}(U) = \pi_0(U) + \pi_3(U), \quad \pi_{12}(U) = \pi_1(U) + \pi_2(U)$$

and obtain

$$\begin{aligned} F^0(U) &= U = \pi_{03}(U) + \pi_{12}(U), \\ F^1(U) &= 3\pi_{03}(U) - \pi_{12}(U), \\ \pi_{03}(U) &= \frac{1}{4}F^0(U) + \frac{1}{4}F^1(U) = \frac{1}{4}U + \frac{1}{4}e^a U e_a, \\ \pi_{12}(U) &= \frac{3}{4}F^0(U) - \frac{1}{4}F^1(U) = \frac{3}{4}U - \frac{1}{4}e^a U e_a, \\ D &= \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}, \quad G = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & -\frac{1}{4} \end{pmatrix}. \end{aligned}$$

## 4 Clifford field vectors and an algebra of $h$ -forms.

**Frame field  $y_a^\mu$ .** A set of  $n$  real vector fields  $y_a^\mu = y_a^\mu(x) \in T^1$  of pseudo-Euclidian space  $\mathbb{R}^{p,q}$  enumerated by the Latin index ( $a = 1, \dots, n$ ) and satisfying

$$y_a^\mu y_b^\nu \eta^{ab} = \eta^{\mu\nu}, \quad \forall x \in \mathbb{R}^{p,q}$$

is called a *frame field*. Using local (that does not depend on  $x$ ) pseudo-orthogonal transformation, we can get another frame field from the frame field  $y_a^\mu$

$$y_a^\mu \rightarrow \hat{y}_a^\mu = q_a^b y_b^\mu,$$

where  $q_a^b = q_a^b(x)$  are smooth functions of  $x \in \mathbb{R}^{p,q}$  and matrix  $Q = Q(x) = \|q_a^b\|$  is such that  $Q \in O(p, q)$  for any  $x$ . It is easy to see that

$$\hat{y}_a^\mu \hat{y}_b^\nu \eta^{ab} = \eta^{\mu\nu}, \quad \forall x \in \mathbb{R}^{p,q}$$

i.e. the set of  $n$  vector fields  $\hat{y}_a^\mu$  is also a frame field.

**Coframe field  $y_\nu^b$ .** A set of  $n$  real covector fields  $y_\nu^b = y_\nu^b(x) \in T_1$  of pseudo-Euclidian space  $\mathbb{R}^{p,q}$  enumerated by the Latin index ( $b = 1, \dots, n$ ) and satisfying

$$y_\mu^a y_\nu^b \eta_{ab} = \eta_{\mu\nu}, \quad \forall x \in \mathbb{R}^{p,q}$$

is called a *coframe field*.

If we have frame field  $y_a^\mu$ , then we can get coframe field using Minkowski matrix:

$$y_\nu^b = \eta^{ab} \eta_{\mu\nu} y_a^\mu.$$

**Clifford field vector  $h^\mu$ .** If  $h^\mu = h^\mu(x)$  are components of vector field with values in  $\mathcal{C}(p, q)$  that satisfy the following relations:

$$h^\mu h^\nu + h^\nu h^\mu = 2\eta^{\mu\nu} e, \quad \mu, \nu = 1, \dots, n \quad (29)$$

for any  $\forall x \in \mathbb{R}^{p,q}$  and the condition

$$\text{Tr}(h^1 \dots h^n) = 0, \quad (30)$$

then the vector  $h^\mu \in \mathcal{C}(p, q)^T$  is called a *Clifford field vector*.

Note that condition (30) holds automatically in the case of even  $n$ , i.e. this condition is necessary for the case of odd  $n$ .

Denote the set of invertible Clifford algebra elements by  $\mathcal{C}^\times(p, q)$ . Note that  $\mathcal{C}^\times(p, q)$  is a Lie group with respect to the Clifford multiplication.

If  $h^\mu$  is a Clifford field vector and  $S \in \mathcal{C}^\times(p, q)^T$  is continuous function, then we can get a couple of new Clifford field vectors using similarity transformation <sup>5</sup>

$$\hat{h}^\mu = \pm S^{-1} h^\mu S.$$

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<sup>5</sup>In the case of even  $n$  it is sufficiently to consider only relation  $\hat{h}^\mu = S^{-1} h^\mu S$  (see [9]).



Example. Let we have a frame field  $y_a^\mu = y_a^\mu(x)$  and a smooth function  $S \in \mathcal{C}^\times(p, q)\mathbb{T}$  with values in the set of invertible Clifford algebra elements. With the help of generators  $e^a$  we get the vector field

$$h^\mu = h^\mu(x) = y_a^\mu S^{-1} e^a S \in \mathcal{C}(p, q)\mathbb{T}^1.$$

It is easy to see that components of this vector field satisfy relations (29) and (30), i.e.  $h^\mu$  is a Clifford field vector.

Components of field vector transform under (orthogonal) changes of coordinates (18) using standard tensor transformation law

$$h^\mu \rightarrow \acute{h}^\mu = p_\nu^\mu h^\nu, \quad (31)$$

where  $P = \|p_\nu^\mu\| \in O(p, q)$ .

With the help of the metric tensor we can raise and lower indices:

$$h_\nu = \eta_{\mu\nu} h^\mu, \quad h^\mu = \eta^{\mu\nu} h_\nu.$$

**Theorem 5** *If  $n = p + q \geq 2$  and  $h^\mu$  is a Clifford field vector, then  $h^\mu \in \mathcal{C}_{\mathbb{S}}(p, q)\mathbb{T}^1$ .*

**Proof.** Let we have a coframe field  $y_\mu^a$ . We define  $n$  elements

$$h^a = y_\mu^a h^\mu \in \mathcal{C}(p, q),$$

satisfying

$$h^a h^b + h^b h^a = 2\eta^{ab} e, \quad \forall a, b = 1, \dots, n.$$

Let  $n = p + q$  be even. We prove that for any  $x \in \mathbb{R}^{p,q}$  we have  $\text{Tr } h^\mu = 0$ . By the generalized Pauli's theorem [13] there exists an invertible element  $S \in \mathcal{C}(p, q)$  (at any  $x \in \mathbb{R}^{p,q}$ ) such that

$$h^a = S^{-1} e^a S, \quad a = 1, \dots, n.$$

So

$$\text{Tr } h^a = \text{Tr}(S^{-1} e^a S) = \text{Tr } e^a = 0, \quad \text{Tr } h^\mu = \text{Tr}(y_\mu^a h^a) = 0.$$

It proves the theorem for the case of even  $n$ .

Let  $n = p + q \geq 3$  be odd. We prove that for any  $x \in \mathbb{R}^{p,q}$  we have  $\text{Tr } h^\mu = 0$  and  $\text{Tr}(e^{1\dots n} h^\mu) = 0$ . By the generalized Pauli's theorem [13] there

exists an invertible element  $S \in \mathcal{C}(p, q)$  (at any  $x \in \mathbb{R}^{p,q}$ ) such that two sets of  $n$  elements  $\{e^a\}$  and  $\{h^a\}$  are related by one of two following formulas:

$$h^a = \epsilon S^{-1} e^a S, \quad a = 1, \dots, n, \quad \epsilon = \pm 1.$$

Then

$$e^{1\dots n} h^a = \epsilon e^{1\dots n} S^{-1} e^a S, \quad a = 1, \dots, n.$$

Note that the element  $e^{1\dots n} e^a$  is an element of rank  $n - 1$ . Therefore  $\text{Tr}(e^{1\dots n} e^a) = 0$  for  $n > 1$ . Since the element  $e^{1\dots n}$  ( $n$  - odd) is from the center of Clifford algebra  $\mathcal{C}(p, q)$ , then for  $n \geq 3$

$$\text{Tr} h^a = \epsilon \text{Tr} e^a = 0, \quad \text{Tr}(e^{1\dots n} h^a) = \epsilon \text{Tr}(e^{1\dots n} e^a) = 0, \quad a = 1, \dots, n.$$

Consequently, for odd  $n \geq 3$  and for any  $x \in \mathbb{R}^{p,q}$  we have

$$\text{Tr} h^\mu = 0, \quad \text{Tr}(e^{1\dots n} h^\mu) = 0, \quad a = 1, \dots, n.$$

It means that  $h^\mu \in \mathcal{C}_{\mathbb{S}}(p, q) T^1$ . ■

**$h$ -forms.** Let we have a covariant skew-symmetric tensor field  $u_{\mu_1 \dots \mu_k} \in T_{[k]}$  of rank  $k$  and a Clifford field vector  $h^\mu \in \mathcal{C}(p, q) T^1$ . We say that the expression

$$\frac{1}{k!} u_{\mu_1 \dots \mu_k} h^{\mu_1} \dots h^{\mu_k} = \sum_{\nu_1 < \dots < \nu_k} u_{\nu_1 \dots \nu_k} h^{\nu_1} \dots h^{\nu_k}$$

is an  $h$ -form of rank  $k$ .

If we have a scalar function  $u = u(x)$  and  $n$  covariant skew-symmetric tensor fields  $u_{\mu_1 \dots \mu_k} \in T_{[k]}$  of ranks  $k = 1, 2, \dots, n$ , then we say that

$$U = ue + \sum_{k=1}^n \frac{1}{k!} u_{\mu_1 \dots \mu_k} h^{\mu_1} \dots h^{\mu_k} = ue + \sum_{k=1}^n \sum_{\nu_1 < \dots < \nu_k} u_{\nu_1 \dots \nu_k} h^{\nu_1} \dots h^{\nu_k} \quad (32)$$

is an  $h$ -form or a *heterogeneous  $h$ -form*.

An  $h$ -form is invariant under orthogonal changes of coordinates (18). Components  $u_{\mu_1 \dots \mu_k}$  of an  $h$ -form are components of covariant skew-symmetric tensor fields of ranks  $k = 0, \dots, n$ .

If we do not pay attention to the difference between tensor (Greek) and nontensor (Latin) indices<sup>6</sup>, then, by relations (29), we can consider components of the field vector  $h^\mu$  as generators of Clifford algebra. A set of  $h$ -forms

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<sup>6</sup>The difference between tensor and nontensor indices appears only when we consider coordinate transformations of pseudo-Euclidian space  $\mathbb{R}^{p,q}$ .

over the field  $\mathbb{F}$  is called the *algebra of  $h$ -forms*  $\mathcal{C}[h]^\mathbb{F}(p, q)$ <sup>7</sup>. We denote the set of  $h$ -forms of rank  $k$  by  $\mathcal{C}[h]_k^\mathbb{F}(p, q)$ . If  $U$  is an  $h$ -form (32) then we denote projections of  $U$  onto  $\mathcal{C}[h]_k^\mathbb{F}(p, q)$  by

$$\pi[h]_k(U), \quad k = 0, 1, \dots, n.$$

To calculate projections  $\pi[h]_k(U)$  we can use method of contractions by components of Clifford field vector using Vandermonde matrix (as in the section 3).

Structure of algebra of  $h$ -forms is considered as a geometrization of structure of Clifford algebra.

Lie algebras generated by Clifford algebra were considered on the page 11. We will use the following Lie algebras generated by the algebra of  $h$ -form:

$$\begin{aligned} &\mathcal{C}[h]_2(p, q), \quad \mathcal{C}[h]_1(p, q) \oplus \mathcal{C}[h]_2(p, q), \quad \mathcal{C}[h]_2(p, q) \oplus \mathcal{C}[h]_3(p, q), \\ &\text{cen}\mathcal{C}[h](p, q), \quad \mathcal{C}[h]_\otimes(p, q), \end{aligned}$$

where  $\text{cen}\mathcal{C}[h](p, q)$  is the center of algebra of  $h$ -forms,  $\mathcal{C}[h]_\otimes(p, q) = \mathcal{C}[h](p, q) \setminus \text{cen}\mathcal{C}[h](p, q)$  is the set of  $h$ -forms with zero projection onto the center of algebra of  $h$ -forms.

Note that

$$\mathcal{C}[h]_\otimes(p, q) \simeq \mathcal{C}_\otimes(p, q),$$

because  $\mathcal{C}[h]_0(p, q) \simeq \mathcal{C}_0(p, q)$  for any natural  $n = p + q$  and  $\mathcal{C}[h]_n(p, q) \simeq \mathcal{C}_n(p, q)$  for any odd  $n$ .

**Tensor fields with values in  $h$ -forms.** Tensor field  $U_{\rho_1 \dots \rho_r}^{\nu_1 \dots \nu_k}$  with values in  $h$ -forms (at point  $x \in \mathbb{R}^{p, q}$ ) belongs to the tensor product of tensor algebra and the algebra of  $h$ -forms. We write  $U_{\rho_1 \dots \rho_r}^{\nu_1 \dots \nu_k} \in \mathcal{C}[h](p, q)T_r^k$ . For example, tensor field  $U_\rho^\nu \in \mathcal{C}[h](p, q)T_1^1$  can be represented as

$$U_\rho^\nu = u_\rho^\nu e + \sum_{k=1}^n \frac{1}{k!} u_{\rho\mu_1 \dots \mu_k}^\nu h^{\mu_1} \dots h^{\mu_k},$$

where  $u_{\rho\mu_1 \dots \mu_k}^\nu = u_{\rho[\mu_1 \dots \mu_k]}^\nu$  are components of  $(1, k+1)$  tensor field which are skew-symmetric w.r.t.  $k$  covariant indices (antisymmetrization is denoted by square brackets).

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<sup>7</sup>In notation  $\mathcal{C}[h]^\mathbb{F}(p, q)$  symbol  $h$  means that the basis is generated by Clifford field vector  $h^\mu$ .

Note that we can consider Clifford field vector  $h^\mu$  as vector with values in  $h$ -forms of rank 1. Actually,

$$h^\mu = \delta_\nu^\mu h^\nu \in \mathcal{C}[h]_1(p, q)T^1,$$

where  $\delta_\nu^\mu$  is Kronecker tensor ( $\delta_r^k = 0$  if  $k \neq r$  and  $\delta_k^k = 1$ ). Also we have

$$h_\mu = \eta_{\mu\nu} h^\nu \in \mathcal{C}[h]_1(p, q)T_1,$$

where  $\eta_{\mu\nu}$  are components of metric tensor of pseudo-Euclidian space  $\mathbb{R}^{p,q}$ .

Note that we also consider tensor fields with values in Lie algebras generated by algebra of  $h$ -form in this paper (see p. 19)).

## 5 Primitive field equation and its gauge symmetry

Consider the equation (system of equations)

$$\partial_\mu h_\rho - [C_\mu, h_\rho] = 0, \quad \mu, \rho = 1, \dots, n, \quad (33)$$

where  $h^\rho \in \mathcal{C}(p, q)T^1$  is an arbitrary Clifford field vector and  $C_\mu = C_\mu(x)$  ( $x \in \mathbb{R}^{p,q}$ ) is covector field with values in  $\mathcal{C}(p, q)$ .

We consider system of equations (33) as a new field equation. This equation is called *a primitive field equation*.

Note that if we have a solution  $C_\mu = C_\mu(x) \in \mathcal{C}(p, q)T_1$  of system of equations (33) and  $\alpha_\mu = \alpha_\mu(x)$  are arbitrary continuous components of covector field with values in center of Clifford algebra, then components  $C_\mu + \alpha_\mu \in \mathcal{C}(p, q)T_1$  also satisfy equation (33).

Therefore it is reasonable to assume that  $C_\mu \in \mathcal{C}_\otimes(p, q)T_1$ .

**Theorem 6** *Let  $h^\nu \in \mathcal{C}_\otimes(p, q)T^1$  be a Clifford field vector and  $C_\mu \in \mathcal{C}_\otimes(p, q)T_1$  satisfy the primitive field equation*

$$\partial_\mu h_\rho - [C_\mu, h_\rho] = 0, \quad \forall \mu, \rho = 1, \dots, n.$$

*Let  $S : \mathbb{R}^{p,q} \rightarrow \mathcal{C}^\times(p, q)$  be a function with values in  $\mathcal{C}^\times(p, q)$  such that*

$$S^{-1} \partial_\mu S \in \mathcal{C}_\otimes(p, q)T_1.$$

Then, the following components of covectors

$$\dot{h}_\rho = S^{-1}h_\rho S \in \mathcal{C}_\otimes(p, q)T_1, \quad \dot{C}_\mu = S^{-1}C_\mu S - S^{-1}\partial_\mu S \in \mathcal{C}_\otimes(p, q)T_1$$

also satisfy the equation

$$\partial_\mu \dot{h}_\rho - [\dot{C}_\mu, \dot{h}_\rho] = 0, \quad \forall \mu, \rho = 1, \dots, n.$$

**Proof.** The condition  $\dot{h}_\rho \in \mathcal{C}_\otimes(p, q)T_1$  holds automatically for every  $S \in \mathcal{C}^\times(p, q)T$  because  $\text{Tr}(S^{-1}h_\rho S) = \text{Tr}(h_\rho)$  in the case of natural  $n$  and  $\pi[h]_n(S^{-1}h_\rho S) = \pi[h]_n(h_\rho)$  in the case of odd  $n$  (see Theorems 1 and 5).

To satisfy the condition  $\dot{C}_\mu \in \mathcal{C}_\otimes(p, q)T_1$  we need functions  $S$  from the class  $\mathbb{S}$

$$\mathbb{S} = \{S \in \mathcal{C}^\times(p, q)T : S^{-1}\partial_\mu S \in \mathcal{C}_\otimes(p, q)T_1\}.$$

Then

$$\begin{aligned} \partial_\mu \dot{h}_\rho - [\dot{C}_\mu, \dot{h}_\rho] &= \partial_\mu(S^{-1}h_\rho S) - (S^{-1}C_\mu S - S^{-1}\partial_\mu S)S^{-1}h_\rho S + \\ &+ S^{-1}h_\rho S(S^{-1}C_\mu S - S^{-1}\partial_\mu S) = \partial_\mu S^{-1}h_\rho S + S^{-1}\partial_\mu h_\rho S + S^{-1}h_\rho \partial_\mu S - \\ &- S^{-1}C_\mu h_\rho S + S^{-1}\partial_\mu S S^{-1}h_\rho S + S^{-1}h_\rho C_\mu S - S^{-1}h_\rho \partial_\mu S = \\ &= S^{-1}(\partial_\mu h_\rho - [C_\mu, h_\rho])S + S^{-1}(S\partial_\mu S^{-1} + \partial_\mu S S^{-1})h_\rho S = 0. \end{aligned}$$

■

Remark. Professor G. A. Alekseev called our attention to the following fact. If we consider elements  $S = S(x)$  as matrices then we can use the well known formula

$$\text{Tr}(S^{-1}\partial_\mu S) = \partial_\mu(\ln(\det S)).$$

By this formula, from the condition  $S^{-1}\partial_\mu S \in \mathcal{C}_\otimes(p, q)T_1$  it follows that  $\det S$  does not depend on  $x \in \mathbb{R}^{p,q}$ . So we may normalized  $S$  and take  $\det S = 1$  or  $\det S = -1$ .

**Theorem 7** Let  $h^\mu \in \mathcal{C}_\otimes(p, q)T_1$  be a Clifford field vector and  $C_\mu \in \mathcal{C}_\otimes(p, q)T_1$  be a covector field. If  $h^\mu$  and  $C_\nu$  are related by equation

$$\partial_\mu h^\nu - [C_\mu, h^\nu] = 0, \quad \forall \mu, \nu = 1, \dots, n,$$

then components of covector field  $C_\mu$  satisfy the conditions

$$\partial_\mu C_\nu - \partial_\nu C_\mu - [C_\mu, C_\nu] = 0, \quad \forall \mu, \nu = 1, \dots, n. \quad (34)$$

Conditions (34) are invariant under the gauge transformation

$$C_\mu \rightarrow \acute{C}_\mu = S^{-1}C_\mu S - S^{-1}\partial_\mu S,$$

where  $S = S(x)$  is a function from  $\mathbb{S}$ , i.e.  $S \in \mathcal{C}^\times(p, q)T$  and  $S^{-1}\partial_\mu S \in \mathcal{C}_{\mathbb{S}}(p, q)T_1$ .

**Proof.** Let us differentiate conditions

$$\partial_\mu h^\lambda = [C_\mu, h^\lambda],$$

and obtain

$$\begin{aligned} \partial_\nu \partial_\mu h^\lambda &= [\partial_\nu C_\mu, h^\lambda] + [C_\mu, \partial_\nu h^\lambda] = \\ &= [\partial_\nu C_\mu, h^\lambda] + [C_\mu, [C_\nu, h^\lambda]], \end{aligned}$$

$$0 = (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)h^\lambda = [\partial_\mu C_\nu - \partial_\nu C_\mu - [C_\mu, C_\nu], h^\lambda]. \quad (35)$$

If an element of Clifford algebra commutes with all generators (with  $h^\mu$ ,  $\mu = 1, \dots, n$  in this case), then this element belongs to the center of Clifford algebra. Therefore, from (35) implies

$$\partial_\mu C_\nu - \partial_\nu C_\mu - [C_\mu, C_\nu] = c_{\mu\nu}e \quad \text{if } n = p + q \text{ is even}$$

and

$$\partial_\mu C_\nu - \partial_\nu C_\mu - [C_\mu, C_\nu] = c_{\mu\nu}e + d_{\mu\nu}e^1 \dots e^n \quad \text{if } n = p + q \text{ is odd,}$$

where  $c_{\mu\nu}$ ,  $d_{\mu\nu}$  are components of tensors of rank 2. Since  $C_\mu \in \mathcal{C}_{\mathbb{S}}(p, q)T_1$ , then (by Theorem 3)  $\partial_\mu C_\nu \in \mathcal{C}_{\mathbb{S}}(p, q)T_2$ . So

$$\partial_\mu C_\nu - \partial_\nu C_\mu - [C_\mu, C_\nu] \in \mathcal{C}_{\mathbb{S}}(p, q)T_2$$

and, hence,  $c_{\mu\nu} = 0, d_{\mu\nu} = 0$ . Equality (34) is proved. Gauge invariance of equality (34) is proved by the formula

$$\partial_\mu \acute{C}_\nu - \partial_\nu \acute{C}_\mu - [\acute{C}_\mu, \acute{C}_\nu] = S^{-1}(\partial_\mu C_\nu - \partial_\nu C_\mu - [C_\mu, C_\nu])S.$$

■

## 6 Solution of the primitive field equation

In this section we find a general solution (up to elements of the center of Clifford algebra) of the primitive field equation

$$\partial_\mu h_\rho - [C_\mu, h_\rho] = 0, \quad \mu, \rho = 1, \dots, n, \quad (36)$$

where  $h^\rho \in \mathcal{C}(p, q)\mathbb{T}^1$  is an arbitrary Clifford field vector and  $C_\mu \in \mathcal{C}_{\mathbb{S}}(p, q)\mathbb{T}_1$ .

**Theorem 8** *Suppose that  $n$  is a natural number and  $C_\mu \in \mathcal{C}_{\mathbb{S}}(p, q)\mathbb{T}_1$ . Then the following two systems of equations are equivalent:*

$$\partial_\mu h_\rho - [C_\mu, h_\rho] = 0 \quad \Leftrightarrow \quad C_\mu = \sum_{k=1}^{\acute{n}} \mu_k \pi[h]_k ((\partial_\mu h^\rho) h_\rho), \quad (37)$$

where  $\acute{n} = n$  for even  $n$ ,  $\acute{n} = n - 1$  for odd  $n$  and

$$\mu_k = \frac{1}{n - (-1)^k(n - 2k)} = \frac{1}{n - \lambda_k}.$$

Remark. Using formulas (26) and (27), we can rewrite general solution (37) of the primitive field equation in the following form (we use contractions and do not use projection operators):

$$C_\mu = \sum_{k=1}^n \mu_k \sum_{l=0}^n b_{kl} F^l((\partial_\mu h^\rho) h_\rho) = \sum_{l=0}^n r_l F^l((\partial_\mu h^\rho) h_\rho), \quad r_l = \sum_{k=1}^n \mu_k b_{kl} \quad (38)$$

in the case of even  $n$  and

$$C_\mu = \sum_{k=1}^{n-1} \mu_k \sum_{l=0}^{\frac{n-1}{2}} g_{kl} F^l((\partial_\mu h^\rho) h_\rho) = \sum_{l=0}^{\frac{n-1}{2}} s_l F^l((\partial_\mu h^\rho) h_\rho), \quad s_l = \sum_{k=1}^{\frac{n-1}{2}} \mu_k g_{kl} \quad (39)$$

in the case of odd  $n$ .

On the page 32 we write explicit formulas for solution of the primitive field equation in the cases of small dimensions  $n = 2, 3, 4$ .

**Proof.** Consider the decomposition of solution  $C_\mu$  of system of equations (36)

$$C_\mu = \sum_{k=0}^n \pi[h]_k(C_\mu), \quad (40)$$

where  $\pi[h]_k(C_\mu) \in \mathcal{C}[h]_k(p, q)T_1$ . Multiply the left side of equation (36) by  $h^\rho$  and consider the corresponding contraction (summation over index  $\rho$ ):

$$h^\rho \partial_\mu h_\rho - h^\rho C_\mu h_\rho + h^\rho h_\rho C_\mu = 0.$$

Using formula (40) and formulas

$$h^\rho h_\rho = ne, \quad h^\rho C_\mu h_\rho = \sum_{k=0}^n h^\rho \pi[h]_k(C_\mu) h_\rho = \sum_{k=0}^n (-1)^k (n - 2k) \pi[h]_k(C_\mu),$$

we obtain

$$\sum_{k=0}^n (n - (-1)^k (n - 2k)) \pi[h]_k(C_\mu) = -h^\rho \partial_\mu h_\rho = (\partial_\mu h^\rho) h_\rho. \quad (41)$$

It easy to see that  $n - (-1)^k (n - 2k) = 0$  holds for  $k = 0, \forall n$  and for  $k = n$ , odd  $n$ . From (41) we obtain required formula (36) for  $C_\mu$ .

Now we shall prove that this expression for  $C_\mu$  satisfies the primitive field equation.

Consider the following contractions  $M_{(-1)^t}^{a,s}(U)$ :

$$M_1^{a,s}(U, h_\nu) = h^{\mu_1} \dots h^{\mu_s} h^{\rho_1} \dots h^{\rho_a} U h_{\rho_a} \dots h_{\rho_1} h_\nu h_{\mu_s} \dots h_{\mu_1},$$

$$M_{-1}^{a,s}(U, h_\nu) = h^{\mu_1} \dots h^{\mu_s} h_\nu h^{\rho_1} \dots h^{\rho_a} U h_{\rho_a} \dots h_{\rho_1} h_{\mu_s} \dots h_{\mu_1}.$$

We contract an arbitrary element  $U \in \mathcal{C}(p, q)$  over  $a + s$  indices. An element  $h_\nu$  is on the right if  $t = 0$  and on the left if  $t = 1$ . The number  $s$  is a distance between  $h_\nu$  and the boundary of expression, the number  $a$  is a distance between  $h_\nu$  and the center of expression.

**Lemma 1** *We have*

$$M_{(-1)^t}^{a,s}(U, h_\nu) = -M_{(-1)^t}^{a-1,s+1}(U, h_\nu) + 2M_{(-1)^{t+1}}^{a-1,s}(U, h_\nu).$$



**Proof.** In the case  $t = 0$  we permute neighboring elements  $h_\nu$  and  $h_{\rho_1}$  using  $h_{\rho_1}h_\nu = -h_\nu h_{\rho_1} + 2\eta_{\nu\rho_1}e$  and obtain 2 another contractions from the statement. In the case  $t = 1$  we use  $h_\nu h^{\rho_1} = -h^{\rho_1}h_\nu + 2\eta_\nu^{\rho_1}e$ . ■

**Lemma 2** *We have*

$$M_{(-1)^t}^{a,s}(U, h_\nu) = \sum_{i=0}^a (-1)^i 2^{a-i} C_a^{a-i} M_{(-1)^{a-i+t}}^{0,i+s}(U, h_\nu).$$

**Proof.** We use the method of mathematical induction (over index  $a$ ). For  $a = 0$  we have  $M_{(-1)^t}^{0,s}(U, h_\nu) = M_{(-1)^t}^{0,s}(U, h_\nu)$ . Suppose that this formula is valid for some  $a$ . Let us prove the validity of this formula for  $a + 1$ . We have

$$\begin{aligned} M_{(-1)^t}^{a+1,s} &= -M_{(-1)^t}^{a,s+1} + 2M_{(-1)^{t+1}}^{a,s} = \\ &= -\sum_{i=0}^a (-1)^i 2^{a-i} C_a^{a-i} M_{(-1)^{a-i+t}}^{0,i+s+1} + 2\sum_{i=0}^a (-1)^i 2^{a-i} C_a^{a-i} M_{(-1)^{a-i+t+1}}^{0,i+s} = \\ &= \sum_{j=1}^{a+1} (-1)^j 2^{a-j+1} C_a^{a-j+1} M_{(-1)^{a-j+1+t}}^{0,j+s} + \sum_{i=0}^a (-1)^i 2^{a-i+1} C_a^{a-i} M_{(-1)^{a-i+t+1}}^{0,i+s} = \\ &= \sum_{i=1}^a (-1)^i 2^{a+1-i} (C_a^{a-i+1} + C_a^{a-i}) M_{(-1)^{a-i+t+1}}^{0,i+s} + (-1)^{a+1} M_{(-1)^t}^{0,a+1-s} + \\ &\quad + 2^{a+1} M_{(-1)^{a+t+1}}^{0,s} = \sum_{i=0}^{a+1} (-1)^i 2^{a+1-i} C_{a+1}^{a+1-i} M_{(-1)^{a+1-i+t}}^{0,i+s}, \end{aligned}$$

where we use  $C_n^{k+1} + C_n^k = C_{n+1}^{k+1}$  and use notation  $M_{(-1)^t}^{a,s}(U, h_\nu) = M_{(-1)^t}^{a,s}$ . ■

We continue the proof of the theorem in the case of even  $n$ . Let us substitute formulas (38) for  $C_\mu$  in the primitive field equation:

$$\partial_\mu h_\nu = \sum_{l=0}^n r_l F^l((\partial_\mu h^\rho)h_\rho)h_\nu - \sum_{l=0}^n r_l h_\nu F^l((\partial_\mu h^\rho)h_\rho).$$

Using Lemmas, we obtain

$$\begin{aligned}
\partial_\mu h_\nu &= \sum_{l=0}^n r_l F^l((\partial_\mu h^\rho)h_\rho)h_\nu - \sum_{l=0}^n r_l h_\nu F^l((\partial_\mu h^\rho)h_\rho) = \\
&= \sum_{l=0}^n r_l (M_1^{l,0}((\partial_\mu h^\rho)h_\rho, h_\nu) - M_{-1}^{l,0}((\partial_\mu h^\rho)h_\rho, h_\nu)) = \\
&= \sum_{l=0}^n r_l \sum_{i=0}^l (-1)^i 2^{l-i} C_l^{l-i} (M_{(-1)^{l-i}}^{0,i}((\partial_\mu h^\rho)h_\rho, h_\nu) - M_{(-1)^{l-i+1}}^{0,i}((\partial_\mu h^\rho)h_\rho, h_\nu)).
\end{aligned}$$

We have

$$\begin{aligned}
&M_{(-1)^{l-i}}^{0,i}((\partial_\mu h^\rho)h_\rho, h_\nu) - M_{(-1)^{l-i+1}}^{0,i}((\partial_\mu h^\rho)h_\rho, h_\nu) = \\
&= (-1)^{l-i} h^{b_1} \dots h^{b_i} ((\partial_\mu h^\rho)h_\rho h_\nu - h_\nu (\partial_\mu h^\rho)h_\rho) h_{b_i} \dots h_{b_1} = \\
&= (-1)^{l-i} F^i((\partial_\mu h^\rho)h_\rho h_\nu - h_\nu (\partial_\mu h^\rho)h_\rho)
\end{aligned}$$

and

$$\begin{aligned}
&(\partial_\mu h^\rho)h_\rho h_\nu - h_\nu (\partial_\mu h^\rho)h_\rho = (\partial_\mu h^\rho)(-h_\nu h_\rho + 2\eta_{\nu\rho}e) + h_\nu h_\rho (\partial_\mu h^\rho) = \\
&= -(\partial_\mu h^\rho)h_\nu h_\rho + 2\partial_\mu h_\nu + (-h_\rho h_\nu + 2\eta_{\rho\nu}e)(\partial_\mu h^\rho) = \\
&= 4\partial_\mu h_\nu - ((\partial_\mu h_\rho)h_\nu h^\rho + h_\rho h_\nu (\partial_\mu h^\rho)) = 4\partial_\mu h_\nu - (\partial_\mu (h_\rho h_\nu h^\rho) - h_\rho \partial_\mu (h_\nu) h^\rho) = \\
&= 4\partial_\mu h_\nu - ((2-n)\partial_\mu h_\nu - h_\rho \partial_\mu (h_\nu) h^\rho) = (2+n)\partial_\mu h_\nu + h_\rho (\partial_\mu h_\nu) h^\rho.
\end{aligned}$$

Then

$$\begin{aligned}
&M_{(-1)^{l-i}}^{0,i}((\partial_\mu h^\rho)h_\rho, h_\nu) - M_{(-1)^{l-i+1}}^{0,i}((\partial_\mu h^\rho)h_\rho, h_\nu) = \\
&= (-1)^{l-i} F^i((2+n)\partial_\mu h_\nu + h_\rho (\partial_\mu h_\nu) h^\rho) = \\
&= (-1)^{l-i} \sum_{m=0}^n \lambda_m^i (2+n+\lambda_m) \pi[h]_m (\partial_\mu h_\nu),
\end{aligned}$$

where  $\lambda_m = (-1)^m (n-2m)$ . So

$$\partial_\mu h_\nu = \sum_{l=0}^n r_l \sum_{i=0}^l (-1)^l 2^{l-i} C_l^{l-i} \sum_{m=0}^n \lambda_m^i (2+n+\lambda_m) \pi[h]_m (\partial_\mu h_\nu),$$

where

$$r_l = \sum_{k=1}^n \mu_k b_{kl} = \sum_{k=1}^n \frac{1}{n-\lambda_k} b_{kl}$$

and  $B = ||b_{kl}||$  is inverse of Vandermonde matrix.

We change index  $j = l - i$  and change the order of summation:

$$\begin{aligned}\partial_\mu h_\nu &= \sum_{m=0}^n (2+n+\lambda_m) \pi[h]_m (\partial_\mu h_\nu) \sum_{k=1}^n \frac{1}{n-\lambda_k} \sum_{l=0}^n b_{kl} (-1)^l \sum_{j=0}^l 2^j C_l^j \lambda_m^{l-j} = \\ &= \sum_{m=0}^n (2+n+\lambda_m) \pi[h]_m (\partial_\mu h_\nu) \sum_{k=1}^n \frac{1}{n-\lambda_k} \sum_{l=0}^n b_{kl} (-1)^l (2+\lambda_m)^l.\end{aligned}$$

Further we consider the sum over  $m$  starting with  $m = 1$  because  $\pi[h]_0 (\partial_\mu h_\nu) = 0$ .

We have

$$-2 - \lambda_m = \lambda_{m+(-1)^{m+1}}, \quad 1 \leq m \leq n. \quad (42)$$

Indeed, in the case of even  $m$  we have

$$-2 - \lambda_m = -2 - (n - 2m) = -2 - n + 2m = -(n - 2(m - 1)) = \lambda_{m-1} = \lambda_{m+(-1)^{m+1}}$$

and in the case of odd  $m$  we have

$$-2 - \lambda_m = -2 + (n - 2m) = -2 + n - 2m = n - 2(m + 1) = \lambda_{m+1} = \lambda_{m+(-1)^{m+1}}.$$

Using (42) and

$$\sum_{l=0}^n b_{kl} (\lambda_a)^l = \delta_{k,a},$$

we obtain

$$\begin{aligned}\partial_\mu h_\nu &= \sum_{m=1}^n (2+n+\lambda_m) \pi[h]_m (\partial_\mu h_\nu) \sum_{k=1}^n \frac{1}{n-\lambda_k} \sum_{l=0}^n b_{kl} (\lambda_{m+(-1)^{m+1}})^l = \\ &= \sum_{m=1}^n (2+n+\lambda_m) \pi[h]_m (\partial_\mu h_\nu) \sum_{k=1}^n \frac{\delta_{k,m+(-1)^{m+1}}}{n-\lambda_k} = \sum_{m=1}^n \frac{(2+n+\lambda_m) \pi[h]_m (\partial_\mu h_\nu)}{n-\lambda_{m+(-1)^{m+1}}} = \\ &= \sum_{m=1}^n \pi[h]_m (\partial_\mu h_\nu).\end{aligned}$$

This completes the proof of theorem for the case of even  $n$ .

Let us prove theorem in the case of odd  $n$ . In this case we have  $\lambda_k = \lambda_{n-k}$ , hence  $\mu_k = \mu_{n-k}$ .

We have

$$C_\mu = \sum_{k=1}^{\frac{n-1}{2}} \mu_k \pi[h]_{k,n-k}((\partial_\mu h^\rho)h_\rho),$$

where  $\pi[h]_{k,n-k}(U) = \pi[h]_k(U) + \pi[h]_{n-k}(U)$ . So

$$C_\mu = \sum_{k=1}^{\frac{n-1}{2}} \mu_k \sum_{l=0}^{\frac{n-1}{2}} g_{kl} F^l((\partial_\mu h^\rho)h_\rho) = \sum_{l=0}^{\frac{n-1}{2}} s_l F^l((\partial_\mu h^\rho)h_\rho), \quad s_l = \sum_{k=1}^{\frac{n-1}{2}} \mu_k g_{kl}.$$

Substitute this expression for  $C_\mu$  in the primitive field equation and obtain

$$\partial_\mu h_\nu = \sum_{l=0}^{\frac{n-1}{2}} s_l F^l((\partial_\mu h^\rho)h_\rho) h_\nu - \sum_{l=0}^{\frac{n-1}{2}} s_l h_\nu F^l((\partial_\mu h^\rho)h_\rho).$$

Similarly to the case of even  $n$  we get

$$\partial_\mu h_\nu = \sum_{l=0}^{\frac{n-1}{2}} s_l \sum_{i=0}^l (-1)^l 2^{l-i} C_l^{l-i} \sum_{m=0}^{\frac{n-1}{2}} \lambda_m^i (2+n+\lambda_m) \pi[h]_{m,n-m}(\partial_\mu h_\nu),$$

where

$$s_l = \sum_{k=1}^{\frac{n-1}{2}} \mu_k g_{kl} = \sum_{k=1}^{\frac{n-1}{2}} \frac{1}{n-\lambda_k} g_{kl}$$

and  $G = ||g_{kl}||$  is inverse of Vandermonde matrix.

Further we consider the sum over  $m$  starting with  $m = 1$  because  $\pi[h]_{0,n}(\partial_\mu h_\nu) = 0$ . We change index  $j = l - i$  and change the order of summation:

$$\begin{aligned} \partial_\mu h_\nu &= \sum_{m=1}^{\frac{n-1}{2}} (2+n+\lambda_m) \pi[h]_{m,n-m}(\partial_\mu h_\nu) \sum_{k=1}^{\frac{n-1}{2}} \frac{1}{n-\lambda_k} \sum_{l=0}^{\frac{n-1}{2}} (-1)^l g_{kl} \sum_{j=0}^l 2^j C_l^j \lambda_m^{l-j} = \\ &= \sum_{m=1}^{\frac{n-1}{2}} (2+n+\lambda_m) \pi[h]_{m,n-m}(\partial_\mu h_\nu) \sum_{k=1}^{\frac{n-1}{2}} \frac{1}{n-\lambda_k} \sum_{l=0}^{\frac{n-1}{2}} (-1)^l g_{kl} (2+\lambda_m)^l. \end{aligned}$$

Using

$$-2 - \lambda_m = \lambda_{m+(-1)^{m+1}}, \quad 1 \leq m \leq n$$

and

$$\sum_{l=0}^{\frac{n-1}{2}} g_{kl} (\lambda_a)^l = \delta_{k,a},$$

we get

$$\begin{aligned} \partial_\mu h_\nu &= \sum_{m=1}^{\frac{n-1}{2}} (2 + n + \lambda_m) \pi[h]_{m,n-m} (\partial_\mu h_\nu) \sum_{k=1}^{\frac{n-1}{2}} \frac{\delta_{k,m+(-1)^{m+1}}}{n - \lambda_k} = \\ &= \sum_{m=1}^{\frac{n-1}{2}} \frac{(2 + n + \lambda_m) \pi[h]_{m,n-m} (\partial_\mu h_\nu)}{n - \lambda_{m+(-1)^{m+1}}} = \sum_{m=1}^{\frac{n-1}{2}} \pi[h]_{m,n-m} (\partial_\mu h_\nu). \end{aligned}$$

So, in the case of odd  $n$  theorem is also proved. ■

## 7 Solutions of Yang-Mills equations

**Yang-Mills equations.** Let  $K$  be a semisimple Lie group and  $L$  be the real Lie algebra of Lie group  $K$ . We denote the set of  $(r, s)$  tensor fields of pseudo-Euclidian space  $\mathbb{R}^{p,q}$  with values in  $L$  by  $LT_s^r$ .

Let  $x \in \mathbb{R}^{p,q}$  and  $B_\mu \in LT_1$ ,  $J^\nu \in LT^1$ ,  $G_{\mu\nu} \in LT_2$ ,  $G_{\mu\nu} = -G_{\nu\mu}$ .  
Equations

$$\begin{aligned} \partial_\mu B_\nu - \partial_\nu B_\mu - [B_\mu, B_\nu] &= G_{\mu\nu}, \\ \partial_\mu G^{\mu\nu} - [B_\mu, G^{\mu\nu}] &= J^\nu \end{aligned} \tag{43}$$

are called *Yang-Mills equations*.

Usually it is assumed that  $B_\mu, G_{\mu\nu}$  are unknowns and  $J^\nu$  is known vector with values in the Lie algebra  $L$ .

Equations (43) define *Yang-Mills field*  $(B_\mu, G_{\mu\nu})$ , where  $B_\mu$  is *potential* and  $G_{\mu\nu}$  is *field strength*. Vector  $J^\nu$  is called *non-Abelian current* (vector  $J^\nu$  is called *current* in the case of Abelian group  $K$ ).

We can substitute components of skew-symmetric tensor  $G_{\mu\nu}$  (the first equation from (43)) for  $G_{\mu\nu}$  in the second equation and obtain one second-order equation for covector potential  $B_\mu$ .

Look at the equation (43) from the other point of view. Let  $B_\mu \in LT_1$  be an arbitrary known covector depending smoothly on  $x \in \mathbb{R}^{p,q}$  with values in  $L$ . Consider the following expression for  $G_{\mu\nu}$

$$G_{\mu\nu} := \partial_\mu B_\nu - \partial_\nu B_\mu - [B_\mu, B_\nu] \quad (44)$$

and expression for  $J^\nu$

$$J^\nu := \partial_\mu G^{\mu\nu} - [B_\mu, G^{\mu\nu}].$$

Then using simple calculations we obtain the following property:

$$\partial_\nu J^\nu - [B_\nu, J^\nu] = 0. \quad (45)$$

This equation is called a *non-Abelian conservation law* (if a group  $K$  is Abelian then we have  $\partial_\nu J^\nu = 0$ , i.e. divergence of vector  $J^\nu$  equals zero).

Hence, non-Abelian conservation law (45) follows from Yang-Mills equations (43).

Let tensor fields  $B_\mu, G_{\mu\nu}, J^\nu$  satisfy Yang-Mills equations (43). Consider the following transformed tensor fields for some element  $U = U(x) \in K$

$$\begin{aligned} \dot{B}_\mu &= U^{-1} B_\mu U - U^{-1} \partial_\mu U, \\ \dot{G}_{\mu\nu} &= U^{-1} G_{\mu\nu} U, \\ \dot{J}^\nu &= U^{-1} J^\nu U. \end{aligned} \quad (46)$$

Then these elements satisfy the same Yang-Mills equations:

$$\begin{aligned} \partial_\mu \dot{B}_\nu - \partial_\nu \dot{B}_\mu - [\dot{B}_\mu, \dot{B}_\nu] &= \dot{G}_{\mu\nu}, \\ \partial_\mu \dot{G}^{\mu\nu} - [\dot{B}_\mu, \dot{G}^{\mu\nu}] &= \dot{J}^\nu, \end{aligned}$$

i.e. equations (43) are invariant under transformations (46). Transformation (46) is called a *gauge transformation* (or a *gauge symmetry*), and group  $K$  is called a *gauge group* of Yang-Mills equations (43).

**Yang-Mills equations with restrictions.** Sometimes we need to consider the situation when the right hand part (vector  $J^\nu$ ) of Yang-Mills equations (43) takes values in some subset (subspace)  $L_1 \subset L$  of the Lie algebra  $L$ . Let  $K_1 \subset K$  be a subgroup of the Lie group  $K$  such that if  $U = U(x) \in K_1$  and  $J^\nu \in L_1 T^1$ , then

$$U^{-1} J^\nu U \in L_1 T^1.$$

System of Yang-Mills equations (43) with additional condition  $J^\nu \in L_1 T^1$  and with gauge transformations from the group  $K_1 \subset K$  is called *system of Yang-Mills equations with restrictions* ( $L_1 \subset L$ ,  $K_1 \subset K$ ).

**Particular solutions of Yang-Mills equations.** System of Yang-Mills equations (43) is a system of nonlinear partial differential equations for unknown tensor fields  $B_\mu(x)$ ,  $G_{\mu\nu}(x)$  and known  $J^\nu(x)$ <sup>8</sup>.

It is interesting to find particular solutions of Yang-Mills equations in the cases when  $J^\nu(x)$  has a special form (for example, dictated by physical arguments). Some aspects of this issue are discussed in [14].

In particular, there is trivial solution of Yang-Mills equations  $B_\mu = 0$ ,  $G_{\mu\nu} = 0$  for  $J^\nu = 0$ . If  $U = U(x) \in K$ , then we can get another (vacuum) solution of Yang-Mills equations using gauge transformation (46):

$$B_\mu = -U^{-1}\partial_\mu U, \quad G_{\mu\nu} = 0, \quad J^\nu = 0.$$

The following theorem describes relation between solutions of the primitive field equation and a class of particular solutions of Yang-Mills equations.

**Theorem 9** *Let  $h^\mu \in \mathcal{C}_\otimes(p, q)T^1$  be a Clifford field vector and  $C_\mu \in \mathcal{C}_\otimes(p, q)T_1$  be covector field that satisfy the primitive field equation (37). Then covector*

$$B_\mu = \sigma h_\mu + C_\mu \in \mathcal{C}_\otimes(p, q)T_1 \quad (47)$$

*is a solution of the following system of Yang-Mills equations:*

$$\partial_\mu B_\nu - \partial_\nu B_\mu - [B_\mu, B_\nu] = G_{\mu\nu}, \quad (48)$$

$$\partial_\mu G^{\mu\nu} - [B_\mu, G^{\mu\nu}] = \varepsilon h^\nu, \quad (49)$$

*where complex (or real) constants  $\sigma, \varepsilon$  are related by the formula*

$$\varepsilon = 4(n-1)\sigma^3.$$

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<sup>8</sup>Accumulated knowledge on the theory of PDEs indicates that for depth analysis of the solutions of the Yang-Mills theory we should consider boundary value problems in some region of space  $\mathbb{R}^{p,q}$ . Formulation of boundary value problem (what boundary conditions and where) depends on the signature of pseudo-Euclidian space  $(p, q)$ . We can expect well-posed problems for Yang-Mills equations primarily in the cases of signatures  $(1, n-1)$ ,  $(n-1, 1)$  (hyperbolic cases) and  $(0, n)$ ,  $(n, 0)$  (elliptic cases).

Remark. Theorem gives a new class of particular solutions (47) of Yang-Mills equations of the form (48), (49). In the case of small dimensions  $n = 2, 3, 4$  expressions  $C_\mu$  from the formula (47) have the following explicit form.

In the case  $n = 2$

$$\begin{aligned} C_\mu &= \sum_{k=1}^2 \mu_k \pi[h]_k ((\partial_\mu h^\rho) h_\rho) = \frac{1}{2} \pi[h]_1 ((\partial_\mu h^\rho) h_\rho) + \frac{1}{4} \pi[h]_2 ((\partial_\mu h^\rho) h_\rho) \\ &= \frac{1}{2} (\partial_\mu h^\rho) h_\rho - \frac{1}{16} h^\alpha (\partial_\mu h^\rho) h_\rho h_\alpha - \frac{3}{32} h^\beta h^\alpha (\partial_\mu h^\rho) h_\rho h_\alpha h_\beta. \end{aligned}$$

In the case  $n = 3$

$$\begin{aligned} C_\mu &= \sum_{k=1}^2 \mu_k \pi[h]_k ((\partial_\mu h^\rho) h_\rho) = \frac{1}{4} \pi[h]_1 ((\partial_\mu h^\rho) h_\rho) + \frac{1}{4} \pi[h]_2 ((\partial_\mu h^\rho) h_\rho) \\ &= \frac{1}{4} \pi[h]_{12} ((\partial_\mu h^\rho) h_\rho) = \frac{3}{16} (\partial_\mu h^\rho) h_\rho - \frac{1}{16} h^\alpha (\partial_\mu h^\rho) h_\rho h_\alpha. \end{aligned}$$

In the case  $n = 4$

$$\begin{aligned} C_\mu &= \sum_{k=1}^4 \mu_k \pi[h]_k ((\partial_\mu h^\rho) h_\rho) = \frac{1}{6} \pi[h]_1 ((\partial_\mu h^\rho) h_\rho) + \frac{1}{4} \pi[h]_2 ((\partial_\mu h^\rho) h_\rho) \\ &\quad + \frac{1}{2} \pi[h]_3 ((\partial_\mu h^\rho) h_\rho) + \frac{1}{8} \pi[h]_4 ((\partial_\mu h^\rho) h_\rho) \\ &= \frac{1}{4} (\partial_\mu h^\rho) h_\rho + \frac{67}{576} h^\alpha (\partial_\mu h^\rho) h_\rho h_\alpha + \frac{73}{2304} h^\beta h^\alpha (\partial_\mu h^\rho) h_\rho h_\alpha h_\beta \\ &\quad - \frac{19}{2304} h^\gamma h^\beta h^\alpha (\partial_\mu h^\rho) h_\rho h_\alpha h_\beta h_\gamma - \frac{25}{9216} h^\delta h^\gamma h^\beta h^\alpha (\partial_\mu h^\rho) h_\rho h_\alpha h_\beta h_\gamma h_\delta. \end{aligned}$$

**Proof.** Components of covector field  $C_\mu$  satisfy (see Theorem 7)

$$\partial_\mu C_\nu - \partial_\nu C_\mu - [C_\mu, C_\nu] = 0, \quad \forall \mu, \nu = 1, \dots, n. \quad (50)$$

Let us substitute  $B_\mu = \sigma h_\mu + C_\mu$  for  $B_\mu$  in the left side of equation (48) and regroup terms. Then we have

$$\begin{aligned} \partial_\mu B_\nu - \partial_\nu B_\mu - [B_\mu, B_\nu] &= -\sigma^2 [h_\mu, h_\nu] \\ &\quad + (\partial_\mu C_\nu - \partial_\nu C_\mu - [C_\mu, C_\nu]) \\ &\quad + \sigma (\partial_\mu h_\nu - [C_\mu, h_\nu]) - \sigma (\partial_\nu h_\mu - [C_\nu, h_\mu]) \\ &= -\sigma^2 [h_\mu, h_\nu]. \end{aligned}$$



Substitute

$$G^{\mu\nu} = -\sigma^2[h^\mu, h^\nu], \quad B_\mu = \sigma h_\mu + C_\mu$$

for  $G^{\mu\nu}$  and  $B_\mu$  in the left side of the second equation (49). Using (33), we get

$$\begin{aligned} \partial_\mu G^{\mu\nu} - [B_\mu, G^{\mu\nu}] &= \sigma^3[h_\mu, [h^\mu, h^\nu]] - \sigma^2(\partial_\mu[h^\mu, h^\nu] - [C_\mu, [h^\mu, h^\nu]]) \\ &= 4(n-1)\sigma^3 h^\nu. \end{aligned}$$

We used the formula

$$[h_\mu, [h^\mu, h^\nu]] = 4(n-1)h^\nu, \quad \nu = 1, \dots, n,$$

which follows from the theorem on generator contraction (see (25)) because

$$\begin{aligned} h_\mu(h^\mu h^\nu - h^\nu h^\mu) &= 2(n-1)h^\nu, \\ (h^\mu h^\nu - h^\nu h^\mu)h_\mu &= -2(n-1)h^\nu. \end{aligned}$$

We also used the formula

$$\partial_\mu[h^\mu, h^\nu] - [C_\mu, [h^\mu, h^\nu]] = 0, \quad (51)$$

which follows from the primitive field equation  $\partial_\mu h^\nu - [C_\mu, h^\nu] = 0$ . Using  $[h^\mu, h^\nu] = 2h^\mu h^\nu - 2\eta^{\mu\nu}e$ , we can see that formula (51) and the formula

$$\partial_\mu(h^\mu h^\nu) - [C_\mu, h^\mu h^\nu] = 0.$$

are equivalent. The validity of the last formula follows from the equalities

$$\begin{aligned} \partial_\mu(h^\mu h^\nu) - [C_\mu, h^\mu h^\nu] &= \\ (\partial_\mu h^\mu)h^\nu + h^\mu \partial_\mu h^\nu - [C_\mu, h^\mu h^\nu] &= \\ (C_\mu h^\mu - h^\mu C_\mu)h^\nu + h^\mu(C_\mu h^\nu - h^\nu C_\mu) - C_\mu h^\mu h^\nu + h^\mu h^\nu C_\mu &= 0. \end{aligned}$$

The theorem is proved. ■

**Theorem 10** *The system of equations (48), (49) and expressions (47) are invariant under gauge transformations*

$$\begin{aligned} B_\mu &\rightarrow \acute{B}_\mu = S^{-1}B_\mu S - S^{-1}\partial_\mu S, \\ G_{\mu\nu} &\rightarrow \acute{G}_{\mu\nu} = S^{-1}G_{\mu\nu}S, \\ h^\nu &\rightarrow \acute{h}^\nu = S^{-1}h^\nu S. \end{aligned}$$

with  $S = S(x) \in \mathbb{S}$ .

**Proof.** The proof is by direct calculation. ■

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